The space of initial conditions and the property of an almost good reduction in discrete Painlevé II equations over finite fields

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Abstract

We investigate the discrete Painlevé equations ($\mathrm{dP_{II}}$ and $\mathrm{qP_{II}}$) over finite fields. We first show that they are well defined by extending the domain according to the theory of the space of initial conditions. Then we treat them over local fields and observe that they have a property that is similar to the good reduction of dynamical systems over finite fields. We can use this property, which can be interpreted as an arithmetic analogue of singularity confinement, to avoid the indeterminacy of the equations over finite fields and to obtain special solutions from those defined originally over fields of characteristic zero.

1 Introduction

In this article, we study the discrete Painlevé equations over finite fields. The discrete Painlevé equations are non-autonomous, integrable mappings which tend to some continuous Painlevé equations for appropriate choices of the continuous limit [1]. When we treate a discrete Painlevé equation over a finite field, we encounter the problem that its time evolution is not always well defined. This problem cannot be solved even if we extend the domain from $\mathbb{F}_q$ to the projective space $\mathbb{PF}_q$, because $\mathbb{PF}_q$ is no longer a field; we cannot determine the values such as $\frac{0}{0}$, $0 \cdot \infty$, $\infty + \infty$ and so on.

There may be two strategies to define the time evolution over a finite field without inconsistencies. One is to reduce the domain so that the time

evolution will not pass the indeterminate states, and the other is to extend the domain so that it can include all the orbits. We take the latter strategy and adopt two approaches.

The first approach is the application of the theory of space of initial conditions developed by Okamoto [2] and Sakai [3]. We show that the dynamics of the equations over finite fields can be well defined in the space of initial conditions.

The second approach we adopt is closely related to the so called arithmetic dynamics, which concerns the dynamics over arithmetic sets such as $\mathbb{Z}$ or $\mathbb{Q}$ or a number field that is of number theoretic interest [4]. In arithmetic dynamics, the change of dynamical properties of polynomial or rational mappings give significant information when reducing them modulo prime numbers. The mapping is said to have good reduction if, roughly speaking, the reduction commutes with the mapping itself [4]. Linear fractional transformations in PGL_2 are typical examples of mappings with a good reduction. Recently bi-rational mappings over finite fields have been investigated in terms of integrability [5]. Since all the orbits are cyclic as far as the mapping is closed over the (projective) space of a finite field and the integrable mapping has a conserved quantity, one can estimate the distribution of orbit length by using Hesse-Weil bounds or numerical calculations. The QRT mappings [6] over finite fields have been studied in detail by choosing the parameter values so that indeterminate points are avoided [7]. They have a good reduction over finite fields.

We prove that, although they do not have a good reduction modulo a prime in general, they have an *almost good reduction*, which is a generalization of good reduction. We first treat in detail the discrete Painlevé II equation ($\mathrm{dP_{II}}$) over finite fields, on which we have briefly reported in our previous letter [8], and then apply the method to the q-discrete Painlevé II equation ($\mathrm{qP_{II}}$). The time evolution of the discrete Painlevé equations can be well defined generically, even when not defined over the projective space of the field. In particular, the reduction from a local field $\mathbb{Q}_p$ to a finite field $\mathbb{F}_p$ is shown to be well defined and is used to obtain some special solutions directly from those over fields of characteristic zero such as $\mathbb{Q}$ or $\mathbb{R}$.

2 The $\mathrm{dP_{II}}$ equation and its space of initial conditions

A discrete Painlevé equation is a non-autonomous and nonlinear second order ordinary difference equation with several parameters. When it is defined over

a finite field, the dependent variable takes only a finite number of values and its time evolution will attain an indeterminate state in many cases for generic values of the parameters and initial conditions. For example, the $\mathrm{dP_{II}}$ equation is defined as

$$u_{n+1} + u_{n-1} = \frac{z_n u_n + a}{1 - u_n^2} \quad (n \in \mathbb{Z}), \tag{1}$$

where $z_n = \delta n + z_0$ and a, δ, z_0 are constant parameters [9]. Let $q = p^k$ for a prime p and a positive integer $k \in \mathbb{Z}_+$. When (1) is defined over a finite field $\mathbb{F}_q$, the dependent variable u_n will eventually take values ± 1 for generic parameters and initial values $(u_0, u_1) \in \mathbb{F}_q^2$, and we cannot proceed to evolve it. If we extend the domain from $\mathbb{F}_q^2$ to $(\mathbb{PF}_q)^2 = (\mathbb{F}_q \cup \{\infty\})^2$, $\mathbb{PF}_q$ is not a field and we cannot define arithmetic operation in (1). To determine its time evolution consistently, we have two choices: One is to restrict the parameters and the initial values to a smaller domain so that the singularities do not appear. The other is to extend the domain on which the equation is defined. In this article, we will adopt the latter approach. It is convenient to rewrite (1) as:

$$\begin{cases} x_{n+1} &= \dfrac{\alpha_n}{1 - x_n} + \dfrac{\beta_n}{1 + x_n} - y_n, \\ y_{n+1} &= x_n, \end{cases} \tag{2}$$

where $\alpha_n := \frac{1}{2}(z_n + a)$, $\beta_n := \frac{1}{2}(-z_n + a)$. Then we can regard (2) as a mapping defined on the domain $\mathbb{F}_q \times \mathbb{F}_q$. To resolve the indeterminacy at $x_n = \pm 1$, we apply the theory of the state of initial conditions developed by Sakai [3]. First we extend the domain to $\mathbb{PF}_q \times \mathbb{PF}_q$, and then blow it up at four points $(x, y) = (\pm 1, \infty), (\infty, \pm 1)$ to obtain the space of initial conditions:

$$\tilde{\Omega}^{(n)} := \mathcal{A}^{(n)}_{(1,\infty)} \cup \mathcal{A}^{(n)}_{(-1,\infty)} \cup \mathcal{A}^{(n)}_{(\infty,1)} \cup \mathcal{A}^{(n)}_{(\infty,-1)}, \tag{3}$$

where $\mathcal{A}^{(n)}_{(1,\infty)}$ is the space obtained from the two dimensional affine space $\mathbb{A}^2$ by blowing up twice as

$$\mathcal{A}^{(n)}_{(1,\infty)} := \Big\{ \big((x-1, y^{-1}), [\xi_1 : \eta_1], [u_1 : v_1]\big) \;\Big|\; \eta_1(x-1) = \xi_1 y^{-1}, (\xi_1 + \alpha_n \eta_1) v_1 = \eta_1 (1-x) u_1 \Big\} \subset \mathbb{A}^2 \times \mathbb{P} \times \mathbb{P}.$$

Similarly,

$$\mathcal{A}^{(n)}_{(-1,\infty)} := \Big\{ \big((x+1, y^{-1}), [\xi_2 : \eta_2], [u_2 : v_2]\big) \ \Big| \\ \eta_2(x+1) = \xi_2 y^{-1}, (-\xi_2 + \beta_n \eta_2) v_2 = \eta_2(1+x) u_2 \Big\},$$

$$\mathcal{A}^{(n)}_{(\infty,1)} := \Big\{ \big((x^{-1}, y-1), [\xi_3 : \eta_3], [u_3 : v_3]\big) \ \Big| \\ \xi_3(y-1) = \eta_3 x^{-1}, (\eta_3 + \alpha_n \xi_3) v_3 = \xi_3(1-y) u_3 \Big\},$$

$$\mathcal{A}^{(n)}_{(\infty,-1)} := \Big\{ \big((x^{-1}, y+1), [\xi_4 : \eta_4], [u_4 : v_4]\big) \ \Big| \\ \xi_4(y+1) = \eta_4 x^{-1}, (-\eta_4 + \beta_n \xi_4) v_3 = \xi_4(1+y) u_4 \Big\}.$$

The bi-rational map (2) is extended to the bijection $\tilde{\phi}_n : \ \tilde{\Omega}^{(n)} \to \tilde{\Omega}^{(n+1)}$ which decomposes as $\tilde{\phi}_n := \iota_n \circ \tilde{\omega}_n$. Here ι_n is a natural isomorphism which gives $\tilde{\Omega}^{(n)} \cong \tilde{\Omega}^{(n+1)}$, that is, on $\mathcal{A}^{(n)}_{(1,\infty)}$ for instance, ι_n is expressed as

$$\begin{aligned} &\big((x-1, y^{-1}), [\xi : \eta], [u : v]\big) \in \mathcal{A}^{(n)}_{(1,\infty)} \\ \to \quad &\big((x-1, y^{-1}), [\xi - \delta/2 \cdot \eta : \eta], [u : v]\big) \in \mathcal{A}^{(n+1)}_{(1,\infty)}. \end{aligned}$$

The automorphism $\tilde{\omega}_n$ on $\tilde{\Omega}^{(n)}$ is induced from (2) and gives the mapping

$$\mathcal{A}^{(n)}_{(1,\infty)} \to \mathcal{A}^{(n)}_{(\infty,1)},\ \mathcal{A}^{(n)}_{(\infty,1)} \to \mathcal{A}^{(n)}_{(-1,\infty)},\ \mathcal{A}^{(n)}_{(-1,\infty)} \to \mathcal{A}^{(n)}_{(\infty,-1)},\ \mathcal{A}^{(n)}_{(\infty,-1)} \to \mathcal{A}^{(n)}_{(1,\infty)}.$$

Under the map $\mathcal{A}^{(n)}_{(1,\infty)} \to \mathcal{A}^{(n)}_{(\infty,1)}$,

$$\begin{array}{lcll} x = 1 & \to & E_2^{(\infty,1)} & u_3 = \left(y - \dfrac{\beta_n}{2}\right) v_3, \\ E_1^{(1,\infty)} & \to & E_1^{(\infty,1)} & [\xi_1 : -\eta_1] = [\alpha_n \xi_3 + \eta_3 : \xi_3], \\ E_2^{(1,\infty)} & \to & y' = 1 & x' = \dfrac{u_1}{v_1} + \dfrac{\beta_n}{2}, \end{array}$$

where $(x,y) \in \mathcal{A}^{(n)}_{(1,\infty)}$, $(x',y') \in \mathcal{A}^{(n)}_{(\infty,1)}$, E_1^{p} and E_2^{p} are the exceptional curves in $\mathcal{A}^{(n)}_{\mathrm{p}}$ obtained by the first blowing up and the second blowing up respectively at the point $\mathrm{p} \in \{(\pm 1, \infty), (\infty, \pm 1)\}$. Similarly under the map $\mathcal{A}^{(n)}_{(\infty,1)} \to \mathcal{A}^{(n)}_{(-1,\infty)}$,

$$\begin{array}{lcll} E_1^{(\infty,1)} & \to & E_1^{(-1,\infty)} & [\xi_3 : \eta_3] = [\eta_2 : (\beta_n - \alpha_n)\eta_2 - \xi_2], \\ E_2^{(\infty,1)} & \to & E_2^{(-1,\infty)} & [u_3 : v_3] = [-\beta_n u_2 : \alpha_n v_2]. \end{array}$$

The mapping on the other points are defined in a similar manner. Note that $\tilde{\omega}_n$ is well-defined in the case $\alpha_n = 0$ or $\beta_n = 0$. In fact, for $\alpha_n = 0$, $E_2^{(1,\infty)}$ and $E_2^{(\infty,1)}$ can be identified with the lines $x = 1$ and $y = 1$ respectively. Therefore we have found that, through the construction of the space of initial conditions, the $\mathrm{dP_{II}}$ equation can be well-defined over finite fields. However there are some unnecessary elements in the space of initial conditions when we consider a finite field, because we are working on a discrete topology and do not need continuity of the map. Let $\tilde{\Omega}^{(n)}$ be the space of initial conditions and $|\tilde{\Omega}^{(n)}|$ be the number of elements of it. For the $\mathrm{dP_{II}}$ equation, we obtain $|\tilde{\Omega}^{(n)}| = (q+1)^2 - 4 + 4(q+1) - 4 + 4(q+1) = q^2 + 10q + 1$, since $\mathbb{PF}_q$ contains $q+1$ elements. However an exceptional curve E_1^{p} is transferred to another exceptional curve $E_1^{\mathrm{p}'}$, and $[1:0] \in E_2^{\mathrm{p}}$ to $[1:0] \in E_2^{\mathrm{p}'}$ or to a point in $E_1^{\mathrm{p}'}$. Hence we can reduce the space of initial conditions $\tilde{\Omega}^{(n)}$ to the minimal space of initial conditions $\Omega^{(n)}$ which is the minimal subset of $\tilde{\Omega}^{(n)}$ including $\mathbb{PF}_q \times \mathbb{PF}_q$, closed under the time evolution. By subtracting unnecessary elements we find $|\Omega^{(n)}| = (q+1)^2 - 4 + 4(q+1) - 4 = q^2 + 6q - 3$. In summary, we obtain the following proposition:

Proposition 1
The domain of the dP_{II} equation over $\mathbb{F}_q$ can be extended to the minimal domain $\Omega^{(n)}$ on which the time evolution at time step n is well defined. Moreover $|\Omega^{(n)}| = q^2 + 6q - 3$.

In figure 1, we show a schematic diagram of the map $\tilde{\omega}_n$ on $\tilde{\Omega}^{(n)}$, and its restriction map $\omega_n := \tilde{\omega}_n|_{\Omega^{(n)}}$ on $\Omega^{(n)}$ with $q = 3$, $\alpha_0 = 1$ and $\beta_0 = 2$. We can also say that the figure 1 is a diagram for the autonomous version of the equation (2) when $\delta = 0$. In the case of $q = 3$, we have $|\tilde{\Omega}^{(n)}| = 40$ and $|\Omega^{(n)}| = 24$.

The above approach is equally valid for the other discrete Painlevé equations and we can define them over finite fields by constructing isomorphisms on the spaces of initial conditions. Thus we conclude that a discrete Painlevé equation can be well defined over a finite field by redefining the initial domain properly. However, for a general nonlinear equation, explicit construction of the space of initial conditions over a finite field is not so straightforward [10] and it will not help us to obtain the explicit solutions. In the next section, we show another extention of the space of initial conditions: we extend it to $\mathbb{Z}_p \times \mathbb{Z}_p$.

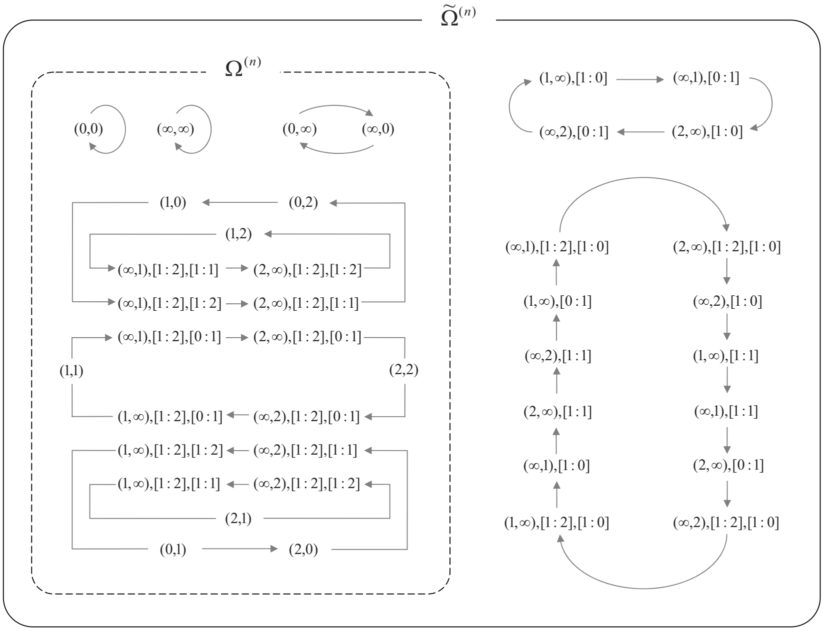

Figure 1: The orbit decomposition of the space of initial conditions $\tilde{\Omega}^{(n)}$ and the reduced one $\Omega^{(n)}$ for $q = 3$.

3 The $\mathrm{dP_{II}}$ equation over a local field and its reduction modulo a prime

Let p be a prime number and for each $x \in \mathbb{Q}$ $(x \neq 0)$ write $x = p^{v_p(x)}\dfrac{u}{v}$ where $v_p(x), u, v \in \mathbb{Z}$ and u and v are coprime integers neither of which is divisible by p. The p-adic norm $|x|_p$ is defined as $|x|_p = p^{-v_p(x)}$. ($|0|_p = 0$.) The local field $\mathbb{Q}_p$ is a completion of $\mathbb{Q}$ with respect to the p-adic norm. It is called the field of p-adic numbers and its subring $\mathbb{Z}_p := \{x \in \mathbb{Q}_p |\ |x|_p \leq 1\}$ is called the ring of p-adic integers [11]. The p-adic norm satisfies a non-archimedean (ultrametric) triangle inequality $|x+y|_p \leq \max[|x|_p, |y|_p]$. Let $\mathfrak{p} = p\mathbb{Z}_p = \{x \in \mathbb{Z}_p |\ v_p(x) \geq 1\}$ be the maximal ideal of $\mathbb{Z}_p$. We define the reduction of x modulo $\mathfrak{p}$ as $\tilde{x}$: $\mathbb{Z}_p \ni x \mapsto \tilde{x} \in \mathbb{Z}_p/\mathfrak{p} \cong \mathbb{F}_p$. Note that the reduction is a ring homomorphism. The reduction is generalized to $\mathbb{Q}_p^\times$:

$$\mathbb{Q}_p^\times \ni x = p^k u \ (u \in \mathbb{Z}_p^\times) \mapsto \begin{cases} 0 & (k > 0) \\ \infty & (k < 0) \\ \tilde{u} & (k = 0) \end{cases} \in \mathbb{PF}_p,$$

which is no longer homomorphic. For a rational map $\phi(x, y) \in \mathbb{Z}_p(x, y)$: $\mathcal{D} \subseteq \mathbb{Z}_p^2 \to \mathbb{Z}_p^2$, defined on some domain $\mathcal{D}$, $\tilde{\phi}(x, y) \in \mathbb{F}_p(x, y)$ is defined as

the map whose coefficients are all reduced. The rational map ϕ is said to have a *good reduction* (modulo $\mathfrak{p}$ on the domain $\mathcal{D}$) if we have $\widetilde{\phi(x,y)} = \tilde{\phi}(\tilde{x},\tilde{y})$ for any $(x,y) \in \mathcal{D}$ [4]. We have defined a generalized notion in our previous letter and have explained its usefulness;

Definition 1 ([8])
A (non-autonomous) rational map ϕ_n: $\mathcal{D} \subseteq \mathbb{Z}_p^2 \to \mathbb{Q}_p$ $(n \in \mathbb{Z})$ is said to have an almost good reduction modulo $\mathfrak{p}$ if there exists a positive integer $m_{\mathrm{p};n}$ for any $\mathrm{p} = (x,y) \in \mathcal{D}$ and time step n such that

$$\widetilde{\phi_n^{m_{\mathrm{p};n}}(x,y)} = \widetilde{\phi_n^{m_{\mathrm{p};n}}}(\tilde{x},\tilde{y}), \tag{4}$$

where $\phi_n^m := \phi_{n+m-1} \circ \phi_{n+m-2} \circ \cdots \circ \phi_n$.

Let us first review some of the findings in [8] in order to see the significance of the notion of *almost good reduction*. Let us consider the mapping Ψ_γ:

$$\begin{cases} x_{n+1} &= \dfrac{ax_n + 1}{x_n^\gamma y_n} \\ y_{n+1} &= x_n \end{cases}, \tag{5}$$

where $a \in \{1, 2, \cdots, p-1\}$ and $\gamma \in \mathbb{Z}_{\geq 0}$ are parameters. The map (5) is known to be integrable if and only if $\gamma = 0, 1, 2$. When $\gamma = 2$, (5) belongs to the QRT family and is integrable in the sense that it has a conserved quantity. We also note that when $\gamma = 0, 1$, (5) is an autonomous version of the q-discrete Painlevé I equation.

Let $\mathcal{D}$ be the domain $\{(x,y) \in \mathbb{Z}_p \,|x \neq 0, y \neq 0\}$, then clearly

$$\widetilde{\Psi_2(x_n,y_n)} = \widetilde{\Psi}_2(\tilde{x}_n,\tilde{y}_n) \qquad \text{for } \tilde{x}_n \neq 0,\ \tilde{y}_n \neq 0.$$

For $(x_n,y_n) \in \mathcal{D}$ with $\tilde{x}_n = 0$ and $\tilde{y}_n \neq 0$, we find that $\widetilde{\Psi_2^k}(\tilde{x}_n = 0, \tilde{y}_n)$ is not defined for $k = 1, 2$, however it is defined if $k = 3$ and we have

$$\widetilde{\Psi_2^3(x_n,y_n)} = \widetilde{\Psi_2^3}(\tilde{x}_n = 0, \tilde{y}_n) = \left(\frac{1}{a^2\tilde{y}}, 0\right).$$

Finally for $\tilde{x}_n = \tilde{y}_n = 0$, we find that $\widetilde{\Psi_2^k}(\tilde{x}_n,\tilde{y}_n)$ is not defined for $k = 1, 2, .., 7$, however

$$\widetilde{\Psi_2^8(x_n,y_n)} = \widetilde{\Psi_2^8}(\tilde{x}_n = 0, \tilde{y}_n = 0) = (0,0)\,.$$

Hence the map Ψ_2 has almost good reduction modulo $\mathfrak{p}$ on $\mathcal{D}$. Note that, in the case $\gamma = 2$ and $a = 0$, if we take

$$f_{2k} := x_{2k}x_{2k-1},\ f_{2k-1} := (x_{2k-1}x_{2k-2})^{-1}$$

(5) turns into the trivial linear mapping $f_{n+1} = f_n$ which has apparently good reduction modulo $\mathfrak{p}$. In a similar manner, we find that Ψ_γ $(\gamma = 0, 1)$ also has almost good reduction modulo $\mathfrak{p}$ on $\mathcal{D}$. On the other hand, for $\gamma \geq 3$ and $\tilde{x}_n = 0$, we easily find that

$$^\forall k \in \mathbb{Z}_{\geq 0}, \quad \widetilde{\Psi^k_\gamma(x_n, y_n)} \neq \widetilde{\Psi^k_\gamma}(\tilde{x}_n = 0, \tilde{y}_n),$$

since the order of p diverges as we iterate the mapping. Thus we have proved the following proposition:

Proposition 2
The rational mapping (5) *has an almost good reduction modulo* $\mathfrak{p}$ *only for* $\gamma = 0, 1, 2$.

Note that having an almost good reduction is equivalent to the integrability of the equation in these examples.

Now let us examine the $\mathrm{dP_{II}}$ (2) over $\mathbb{Q}_p$. We suppose that $p \geq 3$, and redefine the coefficients α_n and β_n so that they are periodic with period p:

$$\alpha_{i+mp} := \frac{(i\delta + z_0 + a + n_\alpha p)}{2}, \ \beta_{i+mp} := \frac{(-i\delta - z_0 + a + n_\beta p)}{2},$$
$$(m \in \mathbb{Z}, \ i \in \{0, 1, 2, \cdots, p-1\}),$$

where the integer n_α (n_β) is chosen such that $0 \in \{\alpha_i\}_{i=0}^{p-1}$ $(0 \in \{\beta_i\}_{i=0}^{p-1})$. As a result, we have $\tilde{\alpha}_n = \frac{\widetilde{n\delta + z_0 + a}}{2}$, $\tilde{\beta}_n = \frac{\widetilde{-n\delta - z_0 + a}}{2}$ and $|\alpha_n|_p, \ |\beta_n|_p \in \{0, 1\}$ for any integer n.

Proposition 3
Under the above assumptions, the $\mathrm{dP_{II}}$ *equation has an almost good reduction modulo* $\mathfrak{p}$ *on* $\mathcal{D} := \{(x, y) \in \mathbb{Z}_p^2 \ | x \neq \pm 1\}$.

Proof We put $(x_{n+1}, y_{n+1}) = \phi_n(x_n, y_n) = \left(\phi_n^{(x)}(x_n, y_n), \phi_n^{(y)}(x_n, y_n)\right)$.

When $\tilde{x}_n \neq \pm 1$, we have

$$\tilde{x}_{n+1} = \frac{\tilde{\alpha}_n}{1 - \tilde{x}_n} + \frac{\tilde{\beta}_n}{1 + \tilde{x}_n} - \tilde{y}_n, \quad \tilde{y}_{n+1} = \tilde{x}_n.$$

Hence $\widetilde{\phi_n(x_n, y_n)} = \tilde{\phi}_n(\tilde{x}_n, \tilde{y}_n)$.

When $\tilde{x}_n = 1$, we can write $x_n = 1 + p^k e$ $(k \in \mathbb{Z}_+, \ |e|_p = 1)$. We have to consider four cases[1]:

[1]Precisely speaking, there are some special cases for $p = 3, 5$ where we have to consider the fact $\alpha_n = \alpha_{n+p}$ or $\beta_n = \beta_{n+p}$. Straightforward calculations prove these exceptional cases.

(i) For $\alpha_n = 0$,

$$\tilde{x}_{n+1} = \tilde{\phi}_n^{(x)}(\tilde{x}_n, \tilde{y}_n) = \widetilde{\left(\frac{\beta_n}{2}\right)} - \tilde{y}_n.$$

Hence we have $\widetilde{\phi_n(x_n, y_n)} = \tilde{\phi}_n(\tilde{x}_n, \tilde{y}_n)$.
(ii) In the case $\alpha_n \neq 0$ and $\beta_{n+2} \neq 0$,

$$\begin{aligned}
x_{n+1} &= -\frac{(\alpha_n - \beta_n)(1 + ep^k) + a}{ep^k(2 + ep^k)} - y_n = -\frac{2\alpha_n + (\alpha_n - \beta_n)ep^k}{ep^k(2 + ep^k)} - y_n, \\
x_{n+2} &= -\frac{\alpha_n^2 + \text{polynomial of } O(p)}{\alpha_n^2 + \text{polynomial of } O(p)}, \\
x_{n+3} &= \frac{\{2\alpha_n y_n + 2\delta\beta_{n+1} + (2-\delta)a\}\alpha_n^3 + \text{polynomial of } O(p)}{2\beta_{n+2}\alpha_n^3 + \text{polynomial of } O(p)},
\end{aligned}$$

Thus we have

$$\tilde{x}_{n+3} = \frac{2\tilde{\alpha}_n\tilde{y}_n + 2\delta\tilde{\beta}_{n+1} + (2-\delta)a}{2\tilde{\beta}_{n+2}}, \quad \tilde{y}_{n+3} = -1,$$

and $\widetilde{\phi_n^3(x_n, y_n)} = \widetilde{\phi_n^3}(\tilde{x}_n, \tilde{y}_n)$.
(iii) In the case $\alpha_n \neq 0$, $\beta_{n+2} = 0$ and $a \neq -\delta$, we have to calculate up to x_{n+5}. After a lengthy calculation we find

$$\tilde{x}_{n+4} = \widetilde{\phi_n^5}^{(y)}(1, \tilde{y}_n) = 1, \text{ and } \tilde{x}_{n+5} = \widetilde{\phi_n^5}^{(x)}(1, \tilde{y}_n) = -\frac{a\delta - (a-\delta)\tilde{y}_n}{a+\delta},$$

and we obtain $\widetilde{\phi_n^5(x_n, y_n)} = \widetilde{\phi_n^5}(\tilde{x}_n, \tilde{y}_n)$.
(iv) Finally, in the case $\alpha_n \neq 0$, $\beta_{n+2} = 0$ and $a = -\delta$ we have to calculate up to x_{n+7}. The result is

$$\tilde{x}_{n+6} = \widetilde{\phi_n^7}^{(y)}(1, \tilde{y}_n) = -1, \text{ and } \tilde{x}_{n+7} = \widetilde{\phi_n^7}^{(x)}(1, \tilde{y}_n) = \frac{1 + 2\tilde{y}_n}{2},$$

and we obtain $\widetilde{\phi_n^7(x_n, y_n)} = \widetilde{\phi_n^7}(\tilde{x}_n, \tilde{y}_n)$. Hence we have proved that the dP$_{\text{II}}$ equation has almost good reduction modulo $\mathfrak{p}$ at $\tilde{x}_n = 1$.

We can proceed in the case $\tilde{x}_n = -1$ in an exactly similar manner and find;
(v) For $\beta_n = 0$, we have $\tilde{x}_{n+1} = \tilde{\phi}_n^{(x)}(\tilde{x}_n = -1, \tilde{y}_n) = \widetilde{\left(\frac{\alpha_n}{2}\right)} - \tilde{y}_n$. Therefore we have $\widetilde{\phi_n(x_n, y_n)} = \widetilde{\phi_n}(\tilde{x}_n, \tilde{y}_n)$.
(vi) In the case $\beta_n \neq 0$ and $\alpha_{n+2} \neq 0$,

$$\widetilde{\phi_n^3(x_n, y_n)} = \widetilde{\phi_n^3}(\tilde{x}_n = -1, \tilde{y}_n) = \left(\frac{a(-2+\delta) - 2\delta\alpha_{n+1} + 2\beta_n\tilde{y}_n}{2\alpha_{n+2}}, 1\right).$$

(vii) In the case $\beta_n \neq 0$, $\alpha_{n+2} = 0$ and $a \neq \delta$,

$$\phi_n^5 \widetilde{(x_n, y_n)} = \widetilde{\phi_n^5}(\tilde{x}_n = -1, \tilde{y}_n) = \left(\frac{a\delta + (a+\delta)\tilde{y}_n}{a - \delta}, -1 \right).$$

(viii) In the case $\beta_n \neq 0$, $\alpha_{n+2} = 0$ and $a = \delta$,

$$\phi_n^7 \widetilde{(x_n, y_n)} = \widetilde{\phi_n^7}(\tilde{x}_n = -1, \tilde{y}_n) = \left(\frac{-1 + 2\tilde{y}_n}{2}, 1 \right).$$

□

From this proposition, the evolution of the dP$_{\mathrm{II}}$ equation (1) over $\mathbb{PF}_p$ can be constructed from the following seven cases which determine $\{u_{n+1}, u_{n+2}, \cdots\}$ from the initial values u_{n-1} and u_n. Note that we can assume that $u_{n-1} \neq \infty$ because all the cases in which the dependent variable u_n becomes ∞ are included below[2]. Here $a = \alpha_n + \beta_n$.

1. For $u_n \in \{2, 3, ..., p-2\}$, or $u_n = 1$ and $\alpha_n = 0$, or $u_n = p-1$ and $\beta_n = 0$,
$$u_{n+1} = \frac{\alpha_n}{1 - u_n} + \frac{\beta_n}{1 + u_n} - u_{n-1}.$$
2. For $u_n = 1$, $\alpha_n \neq 0$ and $\beta_{n+2} \neq 0$,
$$u_{n+1} = \infty,\ u_{n+2} = p - 1,\ u_{n+3} = \frac{2\alpha_n u_{n-1} + 2\delta\beta_{n+1} + (2-\delta)a}{2\beta_{n+2}}.$$
3. For $u_n = 1$, $\alpha_n \neq 0$, $\beta_{n+2} = 0$ and $a + \delta \neq 0$,
$$u_{n+1} = \infty,\ u_{n+2} = p{-}1,\ u_{n+3} = \infty,\ u_{n+4} = 1,\ u_{n+5} = -\frac{a\delta - (a-\delta)u_{n-1}}{a+\delta}.$$
4. For $u_n = 1$, $\alpha_n \neq 0$, $\beta_{n+2} = 0$ and $a + \delta = 0$,
$$u_{n+1} = \infty,\ u_{n+2} = p-1,\ u_{n+3} = \infty,\ u_{n+4} = 1,\ u_{n+5} = \infty,$$
$$u_{n+6} = p - 1,\ u_{n+7} = \frac{1 + 2u_{n-1}}{2}.$$
5. For $u_n = p - 1$, $\beta_n \neq 0$ and $\alpha_{n+2} \neq 0$,
$$u_{n+1} = \infty,\ u_{n+2} = 1,\ u_{n+3} = \frac{a(-2+\delta) - 2\delta\alpha_{n+1} + 2\beta_n u_{n-1}}{2\alpha_{n+2}}.$$

[2]For $p \leq 5$, there are some exceptional cases as shown in the proof of Proposition 3.

6. For $u_n = p-1$, $\beta_n \neq 0$, $\alpha_{n+2} = 0$ and $a \neq \delta$,

$$u_{n+1} = \infty,\ u_{n+2} = 1,\ u_{n+3} = \infty,\ u_{n+4} = p{-}1,\ u_{n+5} = \frac{a\delta + (a+\delta)u_{n-1}}{a-\delta}.$$

7. For $u_n = p-1$, $\beta_n \neq 0$, $\alpha_{n+2} = 0$ and $a = \delta$,

$$\begin{aligned} &u_{n+1} = \infty,\ u_{n+2} = 1,\ u_{n+3} = \infty,\ u_{n+4} = p-1,\ u_{n+5} = \infty,\\ &u_{n+6} = 1,\ u_{n+7} = \frac{-1 + 2u_{n-1}}{2}. \end{aligned}$$

The above approach is closely related to the singularity confinement method which is an effective test to judge the integrability of the given equations [12]. In the proof of the Proposition 3, we have taken $x_n = 1 + ep^k$ and showed that the limit $\lim_{|ep^k|_p \to 0}(x_{n+m}, x_{n+m+1})$ is well defined for some positive integer m. Here ep^k $(k > 0)$ is an alternative in $\mathbb{Q}_p$ for the infinitesimal parameter ϵ in the singularity confinement test in $\mathbb{C}$. Note that p^k $(k > 0)$ is a 'small' number in terms of the p-adic metric. From this observation and propositions 2, 3, we postulate that having almost good reduction in arithmetic mappings is similar to passing the singularity confinement test.

Now we consider special solutions to (1) over $\mathbb{PF}_p$. For the dP$_{\rm II}$ equation over $\mathbb{C}$, rational function solutions have already been obtained [13]. Let N be a positive integer and $\lambda \neq 0$ be a constant. Suppose that $a = -\frac{2(N+1)}{\lambda}$, $\delta = z_0 = \frac{2}{\lambda}$,

$$L_k^{(\nu)}(\lambda) := \begin{cases} \displaystyle\sum_{r=0}^{k} (-1)^r \binom{k+\nu}{k-r} \frac{\lambda^r}{r!} & (k \in \mathbb{Z}_{\geq 0}), \\ 0 & (k \in \mathbb{Z}_{<0}), \end{cases}$$

and

$$\tau_N^n := \det \left| L_{N+1-2i+j}^{(n)}(\lambda) \right|_{1 \leq i,j \leq N}. \tag{6}$$

Then a rational function solution of the dP$_{\rm II}$ equation is given by

$$u_n = \frac{\tau_{N+1}^{n+1}\tau_N^{n-1}}{\tau_{N+1}^n \tau_N^n} - 1. \tag{7}$$

If we deal with the terms in (6) and (7) by arithmetic operations over $\mathbb{F}_p$, we encounter terms such as $\frac{1}{p}$ or $\frac{p}{p}$ and (7) is not well-defined. However, from proposition 3, we find that (7) gives a solution to the dP$_{\rm II}$ equation over $\mathbb{PF}_q$ by the reduction from $\mathbb{Q}(\subset \mathbb{Q}_p)$, as long as the solution avoids the points

$(\tilde{\alpha}_n = 0,\ u_n = 1)$ and $(\tilde{\beta}_n = 0,\ u_n = -1)$, which is equivalent to the solution satisfying

$$\tau_{N+1}^{-N-1}\tau_N^{-N-3} \not\equiv 0,\ \frac{\tau_{N+1}^{N+1}\tau_N^{N-1}}{\tau_{N+1}^{N}\tau_N^{N}} \not\equiv 2, \tag{8}$$

where the superscripts are considered modulo p. Note that $\tau_N^n \equiv \tau_N^{n+p}$ for all integers N and n. In the table below, we give several *rational solutions to the dP$_{II}$ equation* with $N = 3$ and $\lambda = 1$ over $\mathbb{PF}_q$ for $q = 3, 5, 7$ and 11. We see that the period of the solution is p.

p	$\tau_{N+1}^{-N-1}\tau_N^{-N-3}$	$\frac{\tau_{N+1}^{N+1}\tau_N^{N-1}}{\tau_{N+1}^{N}\tau_N^{N}}$	$\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4, \tilde{u}_5, \tilde{u}_6, \tilde{u}_7, \tilde{u}_8, \tilde{u}_9, \tilde{u}_{10}, \dots$
3	∞	∞	$\underbrace{1, 2, \infty}_{\text{period 3}}, 1, 2, \infty, 1, 2, \infty, 1, 2, \infty, 1, 2, \infty, \dots$
5	∞	4	$\underbrace{4, 2, 3, 1, \infty}_{\text{period 5}}, 4, 2, 3, 1, \infty, 4, 2, 3, 1, \infty, 4, \dots$
7	∞	0	$\underbrace{1, \infty, 6, 5, 1, \infty, 6}_{\text{period 7}}, 1, \infty, 6, 5, 1, \infty, 6, 1, \infty, \dots$
11	0	7	$\underbrace{\infty, 1, 6, 1, \infty, 10, \infty, 1, 0, 2, 10}_{\text{period 11}}, \infty, 1, 6, 1, \dots$

We see from the case of $p = 11$ that we may have an appropriate solution even if the condition (8) is not satisfied, although this is not always true. The dP$_{\text{II}}$ equation has linearized solutions also for $\delta = 2a$ [14]. With our new method, we can obtain the corresponding solutions without difficulty.

4 The qP$_{\text{II}}$ equation over a finite field and its special solutions

In this section we study the q-discrete analogue of Painlevé II equation (qP$_{\text{II}}$ equation):

$$(z(q\tau)z(\tau) + 1)(z(\tau)z(q^{-1}\tau) + 1) = \frac{a\tau^2 z(\tau)}{\tau - z(\tau)}, \tag{9}$$

where a and q are parameters [15]. It is also convenient to rewrite (9) as

$$\Phi_n : \begin{cases} x_{n+1} & = \dfrac{a(q^n\tau_0)^2 x_n - (q^n\tau_0 - x_n)(1 + x_n y_n)}{x_n(q^n\tau_0 - x_n)(x_n y_n + 1)}, \\ y_{n+1} & = x_n, \end{cases} \tag{10}$$

where $\tau = q^n\tau_0$. Similarly to the dP$_{\text{II}}$ equation, we can prove the following proposition:

Proposition 4
Suppose that a, q, τ_0 are integers not divisible by p, then the mapping (10) *has an almost good reduction modulo $\mathfrak{p}$ on the domain $\mathcal{D} := \{(x,y) \in \mathbb{Z}_p^2 \,|x \neq 0, x \neq q^n\tau_0\ (n \in \mathbb{Z}), xy+1 \neq 0\}$.*

Proof Let $(x_{n+1}, y_{n+1}) = \Phi_n(x_n, y_n)$. Just like the proof of proposition 3, we have only to examine the cases $\tilde{x}_n = 0, \widetilde{q^n\tau_0}$ and $-\tilde{y}_n^{-1}$. We use the abbreviation $\tilde{q} = q, \tilde{\tau} = \tau, \tilde{a} = a$ for simplicity. By direct computation, we obtain;
(i) If $\tilde{x}_n = 0$ and $-1 + q^2 - aq^2\tau^2 + q^3\tau^2 - q^2\tau\tilde{y}_n \neq 0$,

$$\widetilde{\Phi_n^3(x_n, y_n)} = \widetilde{\Phi_n^3}(\tilde{x}_n = 0, \tilde{y}_n) = \left(\frac{1 - q^2 + aq^2\tau^2 - q^3\tau^2 - aq^4\tau^2 + q^2\tau\tilde{y}_n}{q^2\tau(-1 + q^2 - aq^2\tau^2 + q^3\tau^2 - q^2\tau\tilde{y}_n)}, q^2\tau\right).$$

(ii) If $\tilde{x}_n = 0$ and $-1 + q^2 - aq^2\tau^2 + q^3\tau^2 - q^2\tau\tilde{y}_n = 0$,

$$\widetilde{\Phi_n^5(x_n, y_n)} = \widetilde{\Phi_n^5}(\tilde{x}_n = 0, \tilde{y}_n) = \left(\frac{1 - q^2 + q^7\tau^2 - aq^8\tau^2}{q^4\tau}, 0\right).$$

(iii) If $\tilde{x}_n = \tau$ and $1 + \tau\tilde{y}_n \neq 0$,

$$\begin{aligned}\widetilde{\Phi_n^3(x_n, y_n)} &= \widetilde{\Phi_n^3}(\tilde{x}_n = \tau, \tilde{y}_n) \\ &= \left(\frac{1 - q^2 + (a + q - aq^2)q^2\tau^2 + (1 - q^2)\tau\tilde{y} + (1 - aq)q^3\tau^3\tilde{y}}{q^2\tau(1 + \tau\tilde{y}_n)}, 0\right).\end{aligned}$$

(iv) If $\tilde{x}_n = \tau$ and $1 + \tau\tilde{y}_n = 0$,

$$\widetilde{\Phi_n^7(x_n, y_n)} = \widetilde{\Phi_n^7}(\tilde{x}_n = \tau, \tilde{y}_n) = \left(\frac{1}{aq^{12}\tau^3}, -aq^{12}\tau^3\right).$$

(v) If $\tilde{x_n}\tilde{y}_n + 1 = 0$,

$$\widetilde{\Phi_n^7(x_n, y_n)} = \widetilde{\Phi_n^7}(\tilde{x}_n = -\tilde{y}_n^{-1}, \tilde{y}_n) = \left(-\frac{1}{aq^{12}\tau^4\tilde{y}_n}, aq^{12}\tau^4\tilde{y}_n\right).$$

Thus we complete the proof. □

From this proposition we can define the time evolution of the $\mathrm{qP_{II}}$ equation explicitly just like the $\mathrm{dP_{II}}$ equation in the previous section.

Next we consider special solutions for $\mathrm{qP_{II}}$ equation (9) over $\mathbb{PF}_p$. In [16] it has been proved that (9) over $\mathbb{C}$ with $a = q^{2N+1}$ $(N \in \mathbb{Z})$ is solved by the

functions given by

$$z^{(N)}(\tau)=\begin{cases}\dfrac{g^{(N)}(\tau)g^{(N+1)}(q\tau)}{q^N g^{(N)}(q\tau)g^{(N+1)}(\tau)} & (N\geq 0)\\ \dfrac{g^{(N)}(\tau)g^{(N+1)}(q\tau)}{q^{N+1}g^{(N)}(q\tau)g^{(N+1)}(\tau)} & (N<0)\end{cases}, \tag{11}$$

$$g^{(N)}(\tau)=\begin{cases}\left|w(q^{-i+2j-1}\tau)\right|_{1\leq i,j\leq N} & (N>0)\\ 1 & (N=0)\\ \left|w(q^{i-2j}\tau)\right|_{1\leq i,j\leq -N} & (N<0)\end{cases}, \tag{12}$$

where $w(\tau)$ is a solution of the q-discrete Airy equation:

$$w(q\tau)-\tau w(\tau)+w(q^{-1}\tau)=0. \tag{13}$$

As in the case of the $\mathrm{dP_{II}}$ equation, we can obtain the corresponding solutions to (11) over $\mathbb{PF}_p$ by reduction modulo $\mathfrak{p}$ according to the proposition 4. For that purpose, we have only to solve (13) over $\mathbb{Q}_p$. By elementary computation we obtain:

$$w(q^{n+1}\tau_0)=c_1P_n(\tau_0;q)+c_0P_{n-1}(q\tau_0;q), \tag{14}$$

where $c_0,\ c_1$ are arbitrary constants and $P_n(x;q)$ is defined by the tridiagonal determinant:

$$P_n(x;q):=\begin{vmatrix} qx & -1 & & & \\ -1 & q^2x & -1 & & \Large 0 \\ & \ddots & \ddots & \ddots & \\ & & -1 & q^{n-1}x & -1 \\ \Large 0 & & & -1 & q^nx \end{vmatrix}.$$

The function $P_n(x;q)$ is the polynomial of nth order in x,

$$P_n(x;q)=\sum_{k=0}^{[n/2]}(-1)^k a_{n;k}(q)x^{n-2k},$$

where $a_{n;k}(q)$ are polynomials in q. If we let $i\ll j$ denotes $i<j-1$, and $c(j_1,j_2,...,j_k):=\sum_{r=1}^k(2j_r+1)$, then, we have

$$a_{n;k}=\sum_{1\leq j_1\ll j_2\ll\cdots\ll j_k\leq n-1} q^{n(n+1)/2-c(j_1,j_2,...,j_k)}.$$

Therefore the solution of $\mathrm{qP_{II}}$ equation over $\mathbb{PF}_p$ is obtained by reduction modulo $\mathfrak{p}$ from (11), (12) and (14) over $\mathbb{Q}$ or $\mathbb{Q}_p$.

5 Concluding remarks

In this article we investigated the discrete Painlevé II equations over finite fields. To avoid indeterminacy, we examined two approaches. One is to extend the domain by blowing up at indeterminate points. According to the theory of the space of initial conditions, this approach is possible for all the discrete Painlevé equations. An interesting point is that the space of initial conditions over a finite field can be reduced to a minimal domain because of the discrete topology of the finite field. The other is the reduction modulo prime number from a local field, in particular $\mathbb{Q}_p$. We defined the notion of *almost good reduction* which is an arithmetic analogue of passing the singularity confinement test, and proved that the discrete and q-discrete Painlevé II equations have this property. Thanks to this property, not only the time evolution of the discrete Painlevé equations can be well defined, but also a solution over $\mathbb{Q}$ (or $\mathbb{Q}_p$) can be directly transferred to a solution over $\mathbb{PF}_p$. We presented the special solutions over $\mathbb{PF}_p$. We conjecture that this approach is equally valid in other discrete Painlevé equations and its generalisation [17]. Furthermore, we expect that this 'almost good reduction' criterion can be applied to finding higher order *integrable* mappings in arithmetic dynamics, and that a similar approach is also useful for the investigation of discrete partial difference equations such as soliton equations over finite fields [18, 19]. These problems are currently being investigated.

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